

$\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -INVARIANTS IN IRREDUCIBLE UNIPOTENT REPRESENTATIONS OF $\mathrm{Sp}_{4n}(\mathbb{F}_q)$

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ABSTRACT. We show that for any irreducible representation of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$, the subspace of all its $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -invariants is at most one-dimensional. In terms of Lusztig symbols, we give a complete list of irreducible unipotent representations of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$ which have a nonzero $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -invariant and, in particular, we prove that every irreducible unipotent cuspidal representation has a one-dimensional subspace of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -invariants. As an application, we give an elementary proof of the fact that the unipotent cuspidal representation is defined over \mathbb{Q} , which was proved by Lusztig in [L02].

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1. INTRODUCTION

Let G be a reductive group defined over a field k and θ be an involution of G defined over k . Denote by H the subgroup of all θ -fixed points in G , called a *symmetric subgroup* of G . One of the classical

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problems in harmonic analysis on *symmetric spaces* $G(k)/H(k)$ is to give a formula for the dimension of the space $\text{Hom}_{G(k)}(\pi, \text{Ind}_{H(k)}^{G(k)} \mathbf{1})$ for each irreducible representation π of $G(k)$ and the trivial representation $\mathbf{1}$ of $H(k)$, or equivalently, for the dimension of the space of all $H(k)$ -invariant linear functionals on the space of π . By Frobenius reciprocity, this dimension is equal to the dimension $\dim \pi^{H(k)}$ of the subspace of $H(k)$ -invariants.

For example, let k be a non-archimedean field of characteristic 0 and π be an irreducible supercuspidal representation of $G(k)$ constructed by J.-K. Yu and J.-L. Kim. In [HM08], Hakim and Murnaghan give a formula for the dimension of the space of such invariant linear functionals for all symmetric spaces. This formula reduces the computation of the dimension formula to the “level zero part.” For depth-zero supercuspidal representations, their formula reduces to the analogous problem over the residue field of k . This is one motivation to study such problems over finite fields.

Over a finite field \mathbb{F}_q of odd characteristic p , by convention throughout this paper, we use F to denote the Frobenius map $F: G \rightarrow G$, corresponding to the \mathbb{F}_q -structure of G , and we use the inner product $\langle \cdot, \cdot \rangle_{G^F}$ to denote the dimension of the G^F -invariant Hom-space. Such dimension formulas for symmetric spaces have been studied by Lusztig [L90, L00], Henderson [H03] and others. In [L90], Lusztig gives a dimensional formula for all Deligne-Lusztig virtual representations R_T^λ and for all symmetric subgroups H , that is, for $\langle R_T^\lambda, \text{Ind}_{H^F}^{G^F} \mathbf{1} \rangle_{G^F}$. Further, applying this formula for R_T^λ , Lusztig gives a dimension formula in the case when G^{F^2} and $H = G^F$ in [L00], and Henderson [H03] solves the problem in the case when $G = \text{GL}$ and H is any symmetric subgroup.

The objective of this paper is to consider the case that the symplectic group $G^F = \text{Sp}_{4n}(\mathbb{F}_q)$ and its symmetric subgroup $H^F = \text{Sp}_{2n}(\mathbb{F}_{q^2})$. Let J_{2n} be the skew-symmetric matrix given by

$$J_{2n} = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix},$$

where w_n is the n -by- n permutation matrix with the unit anti-diagonal. Take $G \subset \text{GL}_{4n}$ to be the group preserving the symplectic form defined by J_{4n} . Let θ be the adjoint map $\text{Ad}(\varepsilon)$, where ε is given by

$$\varepsilon = \begin{pmatrix} & I_n & & \\ \tau I_n & & & \\ & & & I_n \\ & & \tau I_n & \end{pmatrix}$$

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and τ is a primitive element in \mathbb{F}_q . Since ε^2 is in the center of the symplectic similitude group GSp_{4n}^F , θ is an involution of G^F and the symmetric subgroup H is isomorphic to $\mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathrm{Sp}_{2n}$, i.e. $H^F = \mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ where \mathbb{F}_{q^2} is the quadratic extension $\mathbb{F}_q(\sqrt{\tau})$ of \mathbb{F}_q .

First, we give an upper bound for the dimension of the subspace of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -invariants in this case. In [Z10], we prove that the symmetric pair $(G(k), H(k))$ defined over non-archimedean field k is a *Gelfand pair*, that is,

$$\dim \mathrm{Hom}_{G(k)}(\pi, \mathrm{Ind}_{H(k)}^{G(k)} \mathbf{1}) \leq 1$$

for all irreducible smooth representations of $G(k)$. Using the calculation in [Z10], we have

Theorem 1.1. *For each irreducible representation π of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$,*

$$\langle \pi, \mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})}^{\mathrm{Sp}_{4n}(\mathbb{F}_q)} \mathbf{1} \rangle \leq 1.$$

Indeed, this can be deduced from $H^F g H^F = H^F g^{-1} H^F$ for all $g \in G^F$ over the finite field \mathbb{F}_q and the calculation whose analogue over non-archimedean local fields was carried out [Z10]. It is easy to check that such a calculation holds for a finite field \mathbb{F}_q of odd characteristic as well. We will omit the details in this paper.

Our goal is to give a complete list of irreducible representations of G^F which have a non-zero H^F -invariant, which are simply called the *distinguished representations*. In this paper, we will focus on *unipotent representations* of G^F , which occur as components of a Deligne-Lusztig virtual representation $R_T^{\mathbf{1}}$, where T is an F -stable maximal torus of G and $\mathbf{1}$ is its trivial character. For a general exposition, see Lusztig [L81, L84].

In this case, unipotent representations are parametrized by symbols of rank n and odd defect. In detail, if G is F -split, the F -stable maximal tori of G^F can be parametrized by conjugacy classes of the Weyl group W of G , we use $R_{T_w}^{\mathbf{1}}$ to denote a Deligne-Lusztig virtual representation. First, unipotent representations can be divided into families by

$$R(c) = \frac{1}{|W|} \mathrm{tr}(w, c) R_{T_w}^{\mathbf{1}}. \quad (1)$$

Here c is a *cell* defined by Lusztig, which is a virtual representation of W . In addition, $R(c)$ is an actual representation of G^F (not necessarily irreducible), with the smallest number of irreducible constituents one can expect from a linear combination of $R_{T_w}^{\mathbf{1}}$'s. In the GL_n case, a cell c is just an irreducible module of the symmetric group S_n and $R(c)$ is an irreducible representation of G^F . There is a one-to-one correspondence between irreducible unipotent representations of GL_n^F and irreducible

S_n -modules. However, this is not true for the other finite groups of Lie type. In the symplectic and odd orthogonal group cases, the irreducible modules of Weyl groups are parametrized by symbols of defect 1 as a subset of symbols of odd defect.

Using Lusztig's formula in [L90], we have a dimension formula for $R_T^{\mathbb{1}}$ and then obtain a W -module Ξ such that for $w \in W$

$$\mathrm{tr}(w, \Xi) = \langle R_{T_w}^{\mathbb{1}}, \mathrm{Ind}_{H^F}^{G^F} \mathbb{1} \rangle.$$

If $G = \mathrm{GL}$, Ξ is always an actual representation of S_n and it is enough to decompose the module Ξ into irreducibles. Once one has the multiplicities of the irreducibles, the multiplicity of the corresponding unipotent representation of G follows. From the above discussion, this is not enough for the other reductive groups and creates difficulties in our case in comparison to the general linear group case. The issue is that even if we know the dimension formula for R_T^λ , there is still more work to be done in order to obtain the dimension formula for all irreducible unipotent representations. In addition to this issue, the W -module Ξ is usually *not* an actual representation in our case and it requires extra work here to decompose Ξ into irreducibles.

Define $\mathcal{U}(\mathrm{Ind}_{H^F}^{G^F} \mathbb{1})$ to be the sub-representation of $\mathrm{Ind}_{H^F}^{G^F} \mathbb{1}$ whose irreducible sub-representations are unipotent representations of G^F . Our purpose is to decompose the representation $\mathcal{U}(\mathrm{Ind}_{H^F}^{G^F} \mathbb{1})$ into irreducible representations of G^F . The strategy is to decompose the virtual representation Ξ_n of W_{2n} , the Weyl group of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$, as

$$\Xi_n = \sum_{r=0}^n \sum_{\beta \vdash n-r} \sum_{i=1}^r (-1)^i \binom{(i, 2r-i) \cdot \beta}{\beta}$$

where pairs of partitions $\binom{\alpha}{\beta}$ parametrize irreducible representations of W_{2n} and $(i, 2r-i) \cdot \beta$ corresponds to an induced product character of $S_{2r} \times S_{2n-2r}$. By the Littlewood-Richardson rule and combinatorial arguments, we divide the irreducibles of Ξ_n into a family of cells $c(Z, \Phi_Z, \hat{\Phi}_Z)$ such that

$$\langle \Xi_n, c(Z, \Phi_Z, \hat{\Phi}_Z) \rangle = 2^d$$

where d is the number of singles in the special symbol Z . By the definition of cells, the associated representation $R(c(Z, \Phi_Z, \hat{\Phi}_Z))$ of G^F has exactly 2^d constituents. Since the multiplicity of each constituent is at most one, every constituent is distinguished in this cell. Therefore, all unipotent representations parametrized by symbols in $c(Z, \Phi_Z, \hat{\Phi}_Z)$ are distinguished and hence we have

Theorem 1.2.

$$\mathcal{U}(\mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})}^{\mathrm{Sp}_{4n}(\mathbb{F}_q)} \mathbf{1}) = \sum_{Z \in \mathcal{S}_n} R(c(Z, \Phi_Z, \hat{\Phi}_Z)).$$

In particular, the unipotent cuspidal representation of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$ has nontrivial $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -invariants.

This result has applications to the theory of irreducible admissible representations of $\mathrm{Sp}_{4n}(k)$ over a non-archimedean field k and to the theory of automorphic representations of Sp_{4n} over number fields. In [Z13], we apply Theorem 1.2 and construct a family of distinguished representations of $\mathrm{Sp}_{4n}(k)$. Further applications including the determination of associated number-theoretic invariants such as L -functions and Arthur parameters and the construction of cuspidal automorphic forms which have a non-zero period integral associated to the symmetric pair will be considered in our forth-coming work.

In this paper, we give an application of Theorem 1.2 to prove the rationality of unipotent representations of symplectic groups. This extends the work of Lusztig [L02, Section 3] to all unipotent cuspidal representations for symplectic groups. We remark that our proof takes an elementary approach without using the Hasse principle. By Theorem 1.2, the unipotent cuspidal representation of G^F is multiplicity-free in the $\mathbb{Q}[G^F]$ -module $\mathrm{Ind}_{H^F}^{G^F} \mathbf{1}$. Since the characters of unipotent representations are \mathbb{Q} -valued, the unipotent cuspidal representation of G^F can be realized by a \mathbb{Q} -module. Hence we obtain

Corollary 1.3. *All unipotent cuspidal representations of symplectic groups over finite fields of odd characteristic are defined over \mathbb{Q} .*

The case of non-unipotent distinguished representations of G^F is also of interest. This will be considered in a sequel to this paper. In that case we need to deal with unipotent representations of *even orthogonal groups* and *general linear groups*. Since the center of symplectic groups is *disconnected*, many arguments given here may be substantially changed.

This paper is organized as follows. In Section 2, for the benefit of the reader, we recall Lusztig's classification theory of unipotent representations of symplectic groups (see Lusztig [L81] or Carter [C93]), and Lusztig's general formula for R_T^λ in [L90]. In Section 3, we decompose the virtual W_{2n} -module Ξ_{2n} into well-understood W_{2n} -modules. In Section 4, we give an explicit combinatorial description of unipotent representations in terms of Lusztig symbols.

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Notation. Denote by S_n the symmetric group of degree n . We also consider w_n as a longest Weyl element of S_n .

Let α be a partition of n with parts $\alpha_1, \alpha_2, \dots, \alpha_m$. Its size, length, and transpose are denoted by $|\alpha|$, $\ell(\alpha)$, and α' . The multiplicity of i as a part of α is denoted by $m_i(\alpha)$ or m_i .

If G is a group and $g \in G$, denote by $Cl_G(g)$ the conjugacy class of g in G and $Z_G(g)$ the centralizer of g in G . Let \mathbb{Z}_n be a cyclic group of order n .

2. UNIPOTENT REPRESENTATIONS OF THE SYMPLECTIC GROUP

In this section, we shall recall the Deligne-Lusztig representations in [DL76] and Lusztig's classification of irreducible representations of symplectic groups over finite fields.

Let G be a connected reductive group defined over a finite field \mathbb{F}_q of odd characteristic p and T be an F -stable maximal torus of G . Let λ be a character of T^F over $\bar{\mathbb{Q}}_l^\times$, where $\bar{\mathbb{Q}}_l$ is an algebraic closure of \mathbb{Q}_l and l is prime to q . For each pair (T, λ) , Deligne and Lusztig [DL76] attached a virtual representation R_T^λ of G^F .

First, there is a bijection between geometric conjugacy classes (see Definition 5.5 in [DL76]) of the pairs (T, λ) and semi-simple conjugacy classes in G^{*F} . Here G^* is the dual group of G defined over \mathbb{F}_q . Further, referring to [L77, Section 7.5], one has a bijection between G^F -conjugacy classes (T, λ) and G^{*F} -conjugacy classes (T', s) , with T' an F -stable maximal torus in G^* and $s \in T'^F$. We also use R_T^s to denote R_T^λ for the corresponding pair (T, λ) .

Let $\mathcal{E}(G)$ be the set of isomorphism classes of irreducible representations of G^F over $\bar{\mathbb{Q}}_l$. Let $\mathcal{E}(G, s)$ be the subset of $\mathcal{E}(G)$, consisting of all irreducible representations ρ such that $\langle \rho, R_T^s \rangle \neq 0$ for some T . One has a partition

$$\mathcal{E}(G) = \coprod_s \mathcal{E}(G, s)$$

where s runs through the set of semi-simple G^{*F} -conjugacy classes in G^{*F} .

An irreducible representation ρ of G^F is *unipotent* if $\langle \rho, R_T^1 \rangle_{G^F} \neq 0$ for some F -stable maximal torus T , that is, ρ is in $\mathcal{E}(G, 1)$. In [L84], Lusztig parametrized all the unipotent representations of reductive groups. We refer to the parametrization of classical groups in Lusztig [L77].

2.1. Unipotent Representations of Symplectic Groups. In this section, we recall the classification of unipotent representations of symplectic groups and Lusztig symbols. See Lusztig [L81] or Carter [C93]. A *symbol* is an unordered pair $\Lambda = (\begin{smallmatrix} S \\ T \end{smallmatrix})$, where S and T are finite sets consisting of non-negative integers. In this paper, the symbol is reduced, i.e. $0 \notin S \cap T$. The *rank* $\text{rk}(\Lambda)$ of Λ is defined by

$$\text{rk}(\Lambda) = \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left\lfloor \left(\frac{\#S + \#T - 1}{2} \right)^2 \right\rfloor,$$

where the square bracket is the greatest integer function. Define by the *defect* $\text{def}(\Lambda) = |\#S - \#T|$. Obviously, the rank of a symbol is a non-negative integer and $\text{rk}(\Lambda) \geq [(\text{def}(\Lambda)/2)^2]$. Lusztig used symbols to parametrize all unipotent representations of G^F as follows.

Proposition 2.1 ([L77, Theorem 8.2]). *For the symplectic group, there exists a one-to-one correspondence between the unipotent representations and the set of symbol classes of rank n and odd defect. Moreover, the symbol Λ with $\text{rk}\Lambda = [(\text{def}(\Lambda)/2)^2]$ corresponds to the cuspidal unipotent representation.*

For example, the unipotent cuspidal representation of $\text{Sp}_{2(d^2+d)}(\mathbb{F}_q)$ corresponds to the symbol

$$\left(\begin{array}{c} 0, 1, 2, \dots, 2d \\ - \end{array} \right).$$

For each symbol Λ of rank n and odd defect we denote by $\rho(\Lambda)$ the corresponding unipotent representation of G^F .

A symbol

$$Z = \left(\begin{array}{c} z_0, z_2, \dots, z_{2m} \\ z_1, z_3, \dots, z_{2m-1} \end{array} \right)$$

of rank n and defect 1 is called *special symbol* if $z_0 \leq z_1 \leq z_2 \leq \dots \leq z_{2m-1} \leq z_{2m}$ holds. Assume that the number of singles in Z (the elements which occur only once) is $2d + 1$. Let Φ be an arrangement of the $2d + 1$ singles in Z into d pairs and one isolated element. An arrangement is called *admissible* if it satisfies the following condition:

there is a pair consisting of consecutive z 's and then the arrangement is still admissible for the new special symbol so-obtained by removing this pair.

Let Φ be an admissible arrangement of the special symbol Z . If Ψ is a subset of Φ , denote by Ψ^* the set of z 's in Ψ occurring in the first row of Z and Ψ_* the set of z 's in Ψ occurring in the second row of Z . Let Z_0 be the set of z 's in Z occurring twice. For any subset $\hat{\Phi}$ of Φ , define a *virtual cell* of W_n , the Weyl group of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, by

$$c(Z, \Phi, \hat{\Phi}) = \sum_{\Psi \subset \Phi} (-1)^{|\hat{\Phi} \cap \Psi|} \left(\frac{Z_0 \prod \Psi^* \prod (\Phi - \Psi)^*}{Z_0 \prod \Psi_* \prod (\Phi - \Psi)_*} \right),$$

where Ψ runs through over all subsets of Φ . This virtual cell $c(Z, \Phi, \hat{\Phi})$ is considered as a virtual representation of W_n . Indeed, each term in $c(Z, \Phi, \hat{\Phi})$ is a symbol of rank n and defect 1 and not special except $\Psi = \emptyset$. There is a one-to-one correspondence between the symbol classes of rank n and defect 1 and the irreducible representations of W_n referring to Section 2.2. Those symbols of defect 1 are also considered as irreducible representations of W_n .

Similar to (1), we have a unipotent representation $R(c(Z, \Phi, \hat{\Phi}))$ of G^F

$$R(c(Z, \Phi, \hat{\Phi})) = \frac{1}{|W_{2n}|} \mathrm{tr}(w, c(Z, \Phi, \hat{\Phi})) R_{T_w}^{\mathbb{1}}.$$

Referring to Lusztig's classification theory, we have

$$R(c(Z, \Phi, \hat{\Phi})) = \sum_{i=1}^{2^d} \rho(\Lambda_i),$$

where Λ_i is uniquely determined by the virtual cell $c(Z, \Phi, \hat{\Phi})$. More details may be found in Lusztig [L81].

Example 2.2. *Let us give an example of unipotent representations of $\mathrm{Sp}_4(\mathbb{F}_q)$. All symbol classes of rank 2 and odd defect are*

$$\left\{ \begin{pmatrix} 2 \\ - \end{pmatrix}, \begin{pmatrix} 1, 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0, 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0, 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0, 1, 2 \\ 1, 2 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 0, 1, 2 \\ - \end{pmatrix} \right\}.$$

The symbols in the first set are of defect 1 and also corresponding to all irreducible representations of W_2 . The symbol in the second set is of defect 3 and corresponding to the unipotent cuspidal representation, that is, θ_{10} in Srinivasan's notation [S68]. Further,

$$R \begin{pmatrix} 2 \\ - \end{pmatrix} \text{ and } R \begin{pmatrix} 0, 1, 2 \\ 1, 2 \end{pmatrix}$$

are irreducible unipotent irreducible representations of $\mathrm{Sp}_4(\mathbb{F})$ and the first one is the trivial representation. For the special symbol $Z = \begin{pmatrix} 0, 2 \\ 1 \end{pmatrix}$, $\Phi_1 = \{(0, 1)\}$ and $\Phi_2 = \{(1, 2)\}$ are the only two admissible arrangements and for instance

$$R(c(Z, \Phi_1, \Phi_1)) = R\begin{pmatrix} 0, 2 \\ 1 \end{pmatrix} - R\begin{pmatrix} 1, 2 \\ 0 \end{pmatrix} = \rho\begin{pmatrix} 0, 1 \\ 2 \end{pmatrix} + \theta_{10}. \quad (2)$$

For additional complementary examples, see Carter [C93, Chapter 13].

Example 2.3. Let $G^F = \mathrm{Sp}_{12}(\mathbb{F}_q)$ and $Z = \begin{pmatrix} 0, 2, 4 \\ 1, 3 \end{pmatrix}$ be a special symbol whose number of singles is 5. Then $\Phi = \{(0, 1), (2, 3)\}$ is an admissible arrangement. We have

$$R(c(Z, \Phi, \Phi)) = \rho\begin{pmatrix} 2, 3, 4 \\ 0, 1 \end{pmatrix} + \rho\begin{pmatrix} 0, 3, 4 \\ 1, 2 \end{pmatrix} + \rho\begin{pmatrix} 0, 1, 2, 3 \\ 4 \end{pmatrix} + \rho\begin{pmatrix} 0, 1, 2, 3, 4 \\ - \end{pmatrix}. \quad (3)$$

2.2. Irreducible Representations of the Weyl Group W_n . In this section, we recall the construction of irreducible representations of W_n in [L81, §2].

Let S_n be the symmetric group of degree n acting on the set $\mathcal{S}_n = \{1, 2, \dots, n\}$. Let α be a partition of n . Denote by $\rho_S(\alpha)$ the irreducible module of S_n whose restriction to $S_{\alpha_1} \times S_{\alpha_2} \times \dots \times S_{\alpha_{\ell(\alpha)}} \subset S_n$ contains the trivial character and whose restriction to $S_{\alpha'_1} \times S_{\alpha'_2} \times \dots \times S_{\alpha'_{\ell(\alpha')}} \subset S_n$ contains the Steinberg character. All irreducible modules of S_n are parameterized by partitions of n in this way. For example, $\rho_S(n)$ is the trivial character and $\rho(1^n)$ is the Steinberg character.

Let W_n act on the set

$$\mathcal{T}_n = \{1, 2, \dots, n, n', \dots, 2', 1'\},$$

and be generated by $(i, i+1)(i', (i+1)')$ for $1 \leq i \leq n-1$ and by (n, n') . Let \mathcal{P}_n be the set of all pairs of partitions $(\alpha; \beta)$ such that $|\alpha| + |\beta| = n$. Then irreducible modules of W_n are parametrized by the pairs in \mathcal{P}_n . Indeed, let $(\alpha; \beta)$ be in \mathcal{P}_n , and $\rho_S(\alpha)$ and $\rho_S(\beta)$ be the irreducible representations of $S_{|\alpha|}$ and $S_{|\beta|}$. Since W_m is isomorphic to $S_m \ltimes (\mathbb{Z}_2)^m$, there are natural liftings $\bar{\rho}_S(\alpha)$ and $\bar{\rho}_S(\beta)$ as representations of $W_{|\alpha|}$ and $W_{|\beta|}$. Define a quadratic character χ of W_n via

$$\chi_n(w) = (-1)^{\#\{w(1), w(2), \dots, w(n)\} \cap \{1', 2', \dots, n'\}}.$$

Then the induced representation

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \mathrm{Ind}_{W_{|\alpha|} \times W_{|\beta|}}^{W_n} \bar{\rho}_S(\alpha) \otimes (\bar{\rho}_S(\beta) \otimes \chi_{|\beta|})$$

is an irreducible representation of W_n corresponding to the ordered pair $(\alpha; \beta)$.

By adding zeroes, we can increase the lengths of α and β at will and order $0 \leq \alpha_i \leq \alpha_{i+1}$ for $1 \leq i \leq m+1$ and $0 \leq \beta_i \leq \beta_{i+1}$ for $1 \leq i \leq m$. By convention, at least one of α_1 and β_1 is not zero. Set $\lambda_i = \alpha_i + i - 1$ for $1 \leq i \leq m+1$ and $\mu_i = \beta_i + i - 1$ for $1 \leq i \leq m$ and then we obtain a symbol of rank n and defect 1

$$\mathcal{L} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \begin{pmatrix} \lambda_1, \lambda_2, \dots, \lambda_{m+1} \\ \mu_1, \mu_2, \dots, \mu_m \end{pmatrix}.$$

Thus we have a one-to-one correspondence between irreducible representations of W_n and symbols of rank n and defect 1.

2.3. Lusztig's Formula for $\langle \text{tr}(\cdot, R_T^\lambda), \text{Ind}_H^G \mathbb{1} \rangle$. For the virtual representations R_T^λ , Lusztig gave a formula in [L90, Theorem 3.3] to compute $\langle \text{tr}(\cdot, R_T^\lambda), \text{Ind}_{H^F}^{G^F} \mathbb{1} \rangle$. We recall the formula in this section. Define

$$\Theta_T = \{f \in G \mid \theta(f^{-1}Tf) = f^{-1}Tf\}.$$

Then T (resp. H) acts on Θ_T by left (resp. right) multiplication. The double cosets $T \backslash \Theta_T / H$ are in one-to-one correspondence with the double cosets $B \backslash G / H$, where B is a Borel subgroup containing T . Let Θ_T^F be the set of F -fixed elements. The double cosets $T^F \backslash \Theta_T^F / H^F$ are also bijective to the G^F -conjugacy classes of (θ, F) -stable maximal tori of G^F .

For any $f \in \Theta_T^F$, define a morphism $\epsilon_{T,\lambda}$ of $(T \cap fHf^{-1})^F$ by

$$\epsilon_{T,\lambda}(t) = (-1)^{\mathbb{F}_q\text{-rank}(Z_G((T \cap fHf^{-1})^\circ)) + \mathbb{F}_q\text{-rank}(Z_G^\circ(t) \cap Z_G((T \cap fHf^{-1})^\circ))}. \quad (4)$$

Then $\epsilon_{T,\lambda}$ is a character and trivial on $((T \cap fHf^{-1})^\circ)^F$. Define

$$\Theta_{T,\lambda}^F = \{f \in \Theta_T^F \mid \lambda_{(T \cap fHf^{-1})^F} = \epsilon_{T,\lambda}\}.$$

The groups T^F and H^F still act on $\Theta_{T,\lambda}^F$ by left and right multiplication.

Proposition 2.4 ([L90, Theorem 3.3]).

$$\langle \text{tr}(\cdot, R_T^\lambda), \text{Ind}_H^G \mathbb{1} \rangle = \sum_{f \in T^F \backslash \Theta_{T,\lambda}^F / H^F} (-1)^{\mathbb{F}_q\text{-rank}(T) + \mathbb{F}_q\text{-rank}(Z_G((T \cap fHf^{-1})^\circ))}$$

Remark that this formula holds for all connected reductive groups and its symmetric subgroups.

3. DECOMPOSITIONS OF W_{2n} -MODULE Ξ_n

3.1. Double Cosets $T \backslash \Theta / H$. Let G be the symplectic group Sp_{4n} . Fix T to be the set of diagonal matrices of G , and B to be the set of upper triangular matrices of G . Then B is an F -stable Borel subgroup of G containing T . Let g be an element of G such that $g^{-1}F(g) \in N_G(T)$. Then gTg^{-1} is also an F -stable maximal torus of G . As G is F -split, the map from gTg^{-1} to the image of $g^{-1}F(g)$ in W gives a bijection between the G^F -conjugacy classes of F -stable maximal tori of G and the conjugacy classes of W . Let g_w be an element in G such $g_w^{-1}F(g_w) = w$. Denote by $T_w = g_wTg_w^{-1}$ the corresponding F -stable maximal torus.

Let ϑ be the map on G defined by $\vartheta(g) = g\theta(g^{-1})$. Then ϑ gives a bijection from G/H to the image $\Im(\vartheta)$ of ϑ . Note that

$$\Im(\vartheta) \subset \{g \in G \mid \theta(g) = g^{-1}\}.$$

Recall that T is a θ -stable maximal torus of G . Let $N(T)$ be the normalizer subgroup of T in G and $\mathcal{N} = N(T) \cap \Im(\vartheta)$. Define a θ -twisted action of T on \mathcal{N} via $t * \omega = t\omega\theta(t)^{-1}$. This θ -twisted action also induces an action of T on $N(T)/T$, which is the ε -conjugation. According to [HW93, Proposition 6.8], G is the disjoint union of the double cosets $B\gamma H$, where $T * \vartheta(\gamma)$ runs through all the orbits of the action of T on \mathcal{N} .

In our case, the Weyl group $N(T)/T$ is W_{2n} . For $x \in N(T)$, decompose x as $x = tv\varepsilon'$ where $t \in T$ and $v \in W_{2n}$ (consider v as a representative in $N(T)$) and

$$\varepsilon' = \begin{pmatrix} & I_n \\ \tau I_n & \\ & \tau^{-1} I_n \\ & & I_n \end{pmatrix} \in \mathrm{Sp}_{4n}(\mathbb{F}_q).$$

Then $\theta(x) = x^{-1}$ if and only if

$$t \cdot vtv^{-1} \cdot \mathrm{diag}\{\tau I_{2n}, I_{2n}\} v \mathrm{diag}\{I_{2n}, \tau^{-1} I_{2n}\} = v^{-1}. \quad (5)$$

Moreover, v is of order 2 in W_{2n} .

Let v be an element of W_{2n} of order 2. Denote by

$$\begin{aligned} \mathcal{S}_v^{(1)} &= \{i \in \mathcal{S}_{2n} \mid v(i) \in \mathcal{S}_{2n}, v(i) > i\}, \\ \mathcal{S}_v^{(2)} &= \{i \in \mathcal{S}_{2n} \mid v(i) = j', v(j) = i', i < j\}, \\ \mathcal{S}_v^{(3)} &= \{i \in \mathcal{S}_{2n} \mid v(i) = i\}, \\ \mathcal{S}_v^{(4)} &= \{i \in \mathcal{S}_{2n} \mid v(i) = i'\}. \end{aligned}$$

Note that the sets $\mathcal{S}_v^{(1)}$, $v(\mathcal{S}_v^{(1)})$, $\mathcal{S}_v^{(2)}$, $v(\mathcal{S}_v^{(2)})'$, $\mathcal{S}_v^{(3)}$, and $\mathcal{S}_v^{(4)}$ are a partition of \mathcal{S}_{2n} , where $i'' = i$. Then, v is of form

$$\prod_{a_i \in \mathcal{S}_v^{(1)}} (a_i, v(a_i))(a'_i, v(a'_i)) \prod_{b_i \in \mathcal{S}_v^{(2)}} (b_i, v(b_i))(v(b'_i), b'_i) \prod_{c_i \in \mathcal{S}_v^{(4)}} (c_i, c'_i).$$

Denote by a_v , b_v and c_v the corresponding product over $\mathcal{S}^{(i)}$ for $i = 1, 2, 4$, respectively. Note that each factor commutes with the others. We choose the following representatives in G^F corresponding to the cycles $(i, j)(i', j')$ and $(i, j)(i', j')(i, i')(j, j')$ respectively by embedding the 4×4 matrices

$$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix},$$

into the $4n \times 4n$ matrices in the (i, j, j', i') -th rows and (i, j, j', i') -th columns, with 1 for the remaining diagonal entries, and zeros for all the remaining non-diagonal entries.

Denote by Ω the set of all pairs (t_v, v) in $T \times W$ satisfying the following conditions:

- $\mathcal{S}_v^{(4)} = \emptyset$;
- $(t_v)_i = \begin{cases} \tau^{-1} & i \in v(\mathcal{S}_v^{(1)}) \\ \pm 1/\sqrt{\tau} & i \in \mathcal{S}_v^{(3)} \\ 1 & i \in \mathcal{S}_{2n} \setminus (v(\mathcal{S}_v^{(1)}) \cup \mathcal{S}_v^{(3)}); \end{cases}$
- $m_v = \frac{1}{2}|\mathcal{S}_v^{(3)}|$, where

$$m_v = \#\{i \in \mathcal{S}_v^{(3)} \mid (t_v)_i = 1/\sqrt{\tau}\}. \quad (6)$$

Let $\gamma(t_v, v)$ be an element in G^F such that $\vartheta(\gamma(t_v, v)) = t_v v \varepsilon'$. Indeed, there exists an element g_1 in GL_{4n} such that $g_1 \varepsilon g_1^{-1} = t_v v \varepsilon' \varepsilon$, since both ε and $t_v v \varepsilon' \varepsilon$ are GL_{4n} -conjugate to $\text{diag}\{\sqrt{\tau}I_{2n}, -\sqrt{\tau}I_{2n}\}$. Since $\varepsilon \in \text{GSp}_{4n}$, there exists an element $\gamma(t_v, v)$ in Sp_{4n} such that $\gamma(t_v, v) \varepsilon \gamma(t_v, v)^{-1} = t_v v \varepsilon' \varepsilon$ and then $\vartheta(\gamma(t_v, v)) = t_v v \varepsilon'$. In addition, $\{t_v v \varepsilon' \mid (t_v, v) \in \Omega\}$ is a subset of \mathcal{N} .

Lemma 3.1. *The set of double cosets $B \backslash G / H$ is in one-to-one correspondence with the set Ω . Moreover, $\{\gamma(t_v, v) \mid (t_v, v) \in \Omega\}$ is a set of double coset representatives.*

Proof. Let $x = t v \varepsilon'$. Since the (i, i) -th entry of $\theta(x)x$ is -1 for $i \in \mathcal{S}_v^{(4)}$, $\theta(x) \neq x^{-1}$ and x is not in \mathcal{N} if $\mathcal{S}_v^{(4)} \neq \emptyset$. Thus for $x \in \mathcal{N}$ we may assume $x = t a_v b_v \varepsilon'$ for some $t = \text{diag}\{t_1, t_2, \dots, t_{2n}, t_{2n}^{-1}, \dots, t_2^{-1}, t_1^{-1}\} \in$

T . Let $t^{(i)}$ for $1 \leq i \leq 3$ be elements in T , and $t_j^{(i)}$ be the (j, j) -th entry given by

$$t_j^{(1)} = \begin{cases} t_j & j \in \mathcal{S}_v^{(1)} \cup v(\mathcal{S}_v^{(1)}) \\ 1 & j \in \mathcal{S}_{2n} \setminus (\mathcal{S}_v^{(1)} \cup v(\mathcal{S}_v^{(1)})), \end{cases} \quad t_j^{(3)} = \begin{cases} t_j & j \in \mathcal{S}_v^{(3)} \\ 1 & j \in \mathcal{S}_{2n} \setminus \mathcal{S}_v^{(3)}, \end{cases}$$

and

$$t_j^{(2)} = \begin{cases} t_j & j \in \mathcal{S}_v^{(2)} \cup v(\mathcal{S}_v^{(2)})' \\ 1 & j \in \mathcal{S}_{2n} \setminus (\mathcal{S}_v^{(2)} \cup v(\mathcal{S}_v^{(2)})'). \end{cases}$$

Then $t = t^{(1)}t^{(2)}t^{(3)}$. Let $d_\tau = \text{diag}\{\tau I_{2n}, \tau^{-1} I_{2n}\}$. We also have the decomposition $d_\tau^{(1)} d_\tau^{(2)} d_\tau^{(3)}$ corresponding to $t^{(i)}$. Further,

$$tvtv^{-1} = ta_vta_v^{-1}b_vtb_v^{-1} = a_vt^{(1)}a_v^{-1}b_vt^{(2)}b_v^{-1}(t^{(3)})^2,$$

and

$$\text{diag}\{\tau I_{2n}, I_{2n}\} w \text{diag}\{I_{2n}, \tau^{-1} I_{2n}\} w = d_\tau^{(1)} a_w^2 b_w^2 d_\tau^{(3)} = d_\tau^{(1)} d_\tau^{(3)}.$$

By $\theta(x) = x^{-1}$ and Equation (5), we have

$$\begin{aligned} I_{2n} &= tvtv^{-1} \text{diag}\{\tau I_{2n}, I_{2n}\} v \text{diag}\{I_{2n}, \tau^{-1} I_{2n}\} v \\ &= d_\tau^{(1)} t^{(1)} a_v t^{(1)} a_v^{-1} \cdot t^{(2)} b_v t^{(2)} b_v^{-1} \cdot (t^{(3)})^2 d_\tau^{(3)}. \end{aligned}$$

Thus

$$t_i = \begin{cases} (\tau t_{v(i)})^{-1} & i \in \mathcal{S}_v^{(1)} \\ t_{v(i)'} & i \in \mathcal{S}_v^{(2)} \\ \pm 1/\sqrt{\tau} & i \in \mathcal{S}_v^{(3)}. \end{cases} \quad (7)$$

Denote by $m_v^+ = \#\{i \in \mathcal{S}_v^{(3)} \mid t_i = 1/\sqrt{\tau}\}$ and $m_v^- = \#\{i \in \mathcal{S}_v^{(3)} \mid t_i = -1/\sqrt{\tau}\}$. Since $x \in \mathcal{N}$, there exists $g \in G$ such that $g\theta(g)^{-1} = x$ and then $g\varepsilon g^{-1} = tv\varepsilon'\varepsilon$. The eigenvalues of ε are $\pm\sqrt{\tau}$ and the dimension of the eigenspaces is $2n$. We have

$$tv\varepsilon'\varepsilon = t^{(1)}a_vt^{(2)}b_vt^{(3)}\text{diag}\{\tau I_{2n}, I_{2n}\},$$

and decompose $\text{diag}\{\tau I_{2n}, I_{2n}\}$ as $(\varepsilon'\varepsilon)^{(1)}(\varepsilon'\varepsilon)^{(2)}(\varepsilon'\varepsilon)^{(3)}$ similarly to the decomposition $t = t^{(1)}t^{(2)}t^{(3)}$.

By Equation (7), the dimensions of the $\pm\sqrt{\tau}$ -eigenspaces corresponding to the matrix $t^{(1)}a_v(\varepsilon'\varepsilon)^{(1)}$ are $2\#\mathcal{S}_v^{(1)}$, the dimensions of the $\pm\sqrt{\tau}$ -eigenspaces corresponding to the matrix $t^{(2)}b_v(\varepsilon'\varepsilon)^{(2)}$ are $2\#\mathcal{S}_v^{(2)}$, and the dimensions of the $\pm\sqrt{\tau}$ -eigenspaces for the matrix $t^{(3)}(\varepsilon'\varepsilon)^{(3)}$ are $2m^+$ and $2m^-$. Thus $m^+ = m^- = \frac{1}{2}\mathcal{S}_v^{(3)}$.

Let $h \in T$. The θ -twisted action $h * x$ is given by

$$hx\theta(h)^{-1} = h^{(1)}t^{(1)}(a_v(h^{(1)})^{-1}a_v^{-1})(h^{(2)})^{-1}t^{(2)}(c_vh^{(2)}c_v^{-1})v\varepsilon'. \quad (8)$$

Choose the element $h \in T$ defined by

$$h_i = \begin{cases} t_i^{-1} & i \in \mathcal{S}_v^{(1)} \cup \mathcal{S}_v^{(2)} \\ 1 & i \in \mathcal{S}_{2n} \setminus (\mathcal{S}_v^{(1)} \cup \mathcal{S}_v^{(2)}). \end{cases}$$

By Equation (7) and (11), $h * x = t_v v \varepsilon'$. Each T -orbit on \mathcal{N} has a nontrivial intersection with $\{t_v v \varepsilon' \mid (t_v, v) \in \Omega\}$. In addition, by the above discussion, for the different pairs (t_{v_1}, v_1) and (t_{v_2}, v_2) , the elements $t_{v_1} v_1 \varepsilon'$ and $t_{v_2} v_2 \varepsilon'$ are in different T -orbits. This completes the proof of the lemma. \square

Corollary 3.2. *The set $\{\gamma(t_v, v) \mid (t_v, v) \in \Omega\}$ is a set of coset representatives for $T \backslash \Theta / H$. Furthermore, $\Omega_w := \{g_w \gamma(t_v, v) \mid (t_v, v) \in \Omega\}$ is a set of coset representatives for $T_w \backslash \Theta_{T_w} / H$.*

Since T_w and H are connected, and θ and F commute, the action of F on G induces an action on the double cosets $T_w \backslash \Theta_{T_w} / H$. In addition, since the group $T_w \cap \gamma H \gamma^{-1}$ are connected for all γ in $T_w \backslash \Theta_{T_w} / H$, the action of F on $T_w \backslash \Theta_{T_w} / H$ is a permutation on the representatives Ω_w . Let Ω_w^F be the subset consisting of all fixed orbits under the action of F . Then

$$T_w^F \backslash \Theta_{T_w}^F / H^F \leftrightarrow \Omega_w^F. \quad (9)$$

Define

$$\Omega_{\pm}^w = \{(t_v, v) \in \Omega \mid v \in Z_W(w) \text{ and } w(t_v)_i w^{-1} = -(t_v)_i \text{ for } i \in \mathcal{S}_v^{(3)}\}.$$

Lemma 3.3. *If T_w is an F -stable maximal torus corresponding to w , then $\Omega_w^F = \Omega_{\pm}^w$, i.e. there is a bijection*

$$T_w^F \backslash \Theta_{T_w}^F / H^F \leftrightarrow \Omega_{\pm}^w.$$

Proof. $F(g_w \gamma(t_v, v))$ is in the double coset $T_w \cdot g_w \gamma(t_{v'}, v') \cdot H$ if and only if for some $t \in T$

$$wF(t_v) v \varepsilon' \varepsilon w^{-1} = t t_v v \varepsilon' \varepsilon t^{-1}$$

$$\iff wF(t_v) \text{diag}\{\tau I_{2n}, I_{2n}\}^v w^{-1} \cdot w v w^{-1} = t_v \text{diag}\{\tau I_{2n}, I_{2n}\}^v \cdot t v t^{-1}$$

where $\text{diag}\{\tau I_{2n}, I_{2n}\}^v = v \cdot \text{diag}\{\tau I_{2n}, I_{2n}\} \cdot v^{-1}$. Then v commutes with w in W . Since $v^2 = I_{4n}$ and $v \in Z_W(w)$, $w v w^{-1}$ is T^F -conjugate to v and $t v t^{-1} \cdot w v^{-1} w^{-1} = t_0$ for some t_0 in the connected component of the v -split torus $\{t \in T \mid v t v^{-1} = t^{-1}\}$. The statement is equivalent to

$$wF(t_v) \text{diag}\{\tau I_{2n}, I_{2n}\}^v w^{-1} = t_v \text{diag}\{\tau I_{2n}, I_{2n}\}^v \cdot t_0$$

for some t_0 in the connected component of v -split torus. Let $g = t_v \text{diag}\{\tau I_{2n}, I_{2n}\}^v$. It is enough to prove that $wF(g) w^{-1} g^{-1}$ is in the

connected component of v -split torus. Since $vgv^{-1}g = \tau I_{4n}$, the existence of t_0 is equivalent to $w(t_v)_i w^{-1} = -(t_v)_i$ for each $i \in \mathcal{S}_v^{(3)}$ under the assumption $v \in Z_W(w)$. Then this lemma follows. \square

According to [C93, Theorem 3.5.6], for a connected reductive group G , if its derived group is simply connected, then $Z_G(s)$ is connected for any semi-simple element s . Since G^F is the symplectic group $\mathrm{Sp}_{4n}(\mathbb{F}_q)$, $Z_G(t)$ and $T \cap fHf^{-1}$ in the definition (4) are connected. Thus $\epsilon_{T,f}$ is trivial. If the character λ is trivial, then $T^F \backslash \Theta_{T,\lambda}^F / H^F$ is the same as $T^F \backslash \Theta_T^F / H^F$.

Example 3.4. Let G be Sp_4 . By the lemmas above, it is easy to check that Ω contains 4 elements. If (t_v, v) is in Ω , then v is the identity e or $(1, 2)(1', 2')$ or $(1, 2')(1', 2)$. If $v = e$, there are two choices of t_v , which are $\pm \mathrm{diag}\{1/\sqrt{\tau}, -1/\sqrt{\tau}, -\sqrt{\tau}, \sqrt{\tau}\}$. If $v \neq e$, t_v is trivial. Hence we simply use $\{(1, 2), (1, 2'), (+, -), (-, +)\}$ to parametrize these 4 double cosets. We give the centralizer $Z_G(T_w \cap fHf^{-1})$ for each F -stable torus T_w and the associated double cosets.

	$(1^2; 0)$	$(2; 0)$	$(1; 1)$	$(0; 2)$	$(0; 1^2)$
$(T_w)^F$	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$\mathbb{F}_{q^2}^\times$	$\mathbb{F}_q^\times \times \mathbb{F}_{q^2}^1$	$\mathbb{F}_{q^4}^1$	$\mathbb{F}_{q^2}^1 \times \mathbb{F}_{q^2}^1$
$(1, 2)$	$\mathrm{GL}_2(\mathbb{F}_q)$	$\mathrm{GL}_2(\mathbb{F}_q)$	—	—	$\mathrm{U}_2(J, \mathbb{F}_{q^2})$
$(+, -)$	—	$\mathbb{F}_{q^2}^\times$	—	$\mathbb{F}_{q^4}^1$	—
$(-, +)$	—	$\mathbb{F}_{q^2}^\times$	—	$\mathbb{F}_{q^4}^1$	—
$(1, 2')$	$\mathrm{GL}_2(\mathbb{F}_q)$	$\mathrm{U}_2(\mathbb{F}_{q^2})$	—	—	$\mathrm{U}_2(J, \mathbb{F}_{q^2})$
$R_w^\mathbb{1}$	2	2	0	2	-2

The first row lists the pairs of partitions in \mathcal{P}_2 parametrizing the G^F -conjugacy classes of F -stable maximal tori. The last row is the dimension formula $< R_w^\mathbb{1}, \mathrm{Ind}_{H^F}^{G^F} \mathbb{1} >$. The symbol ‘—’ means that the corresponding double coset is not F -stable and not in the set Ω_\pm^w .

3.2. Decomposition of Ξ_n . There is a one-to-one correspondence between pairs of partitions $(\alpha; \beta)$ such that $|\alpha| + |\beta| = n$ and the conjugacy classes in W_n . If i and j are parts of α and β respectively, we map i and j to Coxter elements in S_i and W_j respectively, which extends to a bijection between pairs of partitions $(\alpha; \beta)$ in \mathcal{P}_n and the conjugacy classes in W_n . Denote by $Cl_{W_{|\alpha|+|\beta|}}(\alpha; \beta)$ the conjugacy class corresponding to $(\alpha; \beta)$. If w is in W_{2n} of cycle-type $(\alpha; \beta)$, then

$$T_w^F = \prod_i \mathbb{F}_{q^i}^{\times m_i(\alpha)} \times \prod_i (\mathbb{F}_{q^{2i}}^1)^{m_i(\beta)} \text{ and } \mathbb{F}_q\text{-rank}(T_w) = \ell(\alpha),$$

where $\mathbb{F}_{q^{2i}}^1$ is the unitary group $\mathrm{U}_1(\mathbb{F}_{q^{2i}})$.

Let $(\alpha; \beta)$ be in \mathcal{P}_{2n} . Let $I_1(\alpha) = \{(1 \ 1'), (2 \ 2'), \dots, (\ell(\alpha) \ \ell(\alpha)')\}$ where $(i \ i')$ is a permutation in W_{2n} , and $I_2(\beta) = \{1, 2, \dots, \ell(\beta)\}$ be sets indexing the parts of α and β respectively. Define

$$P_1(\alpha) = \{((i \ i'), j) \mid (i \ i') \in I_1(\alpha) \text{ and } j \in \mathbb{Z}_{\alpha_i}\}$$

and

$$P_2(\beta) = \{(i, j) \mid i \in I_2(\beta) \text{ and } j \in \mathbb{Z}_{2\beta_i}\}.$$

Then w is a permutation of $P_1(\alpha)$ and $P_2(\beta)$ by $w(i, j) = (i, j + 1)$.

For each $(t_v, v) \in \Omega_{\pm}^w$, v induces permutations of $P_1(\alpha)$ and $P_2(\beta)$ of order 2, denoted by \bar{v} and \tilde{v} respectively. There are unique elements $j_1(v, i)$ and $j_2(v, i)$ such that $v(i, j) = (\bar{v}(i), j + j_1(v, i))$ for $(i, j) \in P_1(\alpha)$ and $v(i, j) = (\tilde{v}(i), j + j_2(v, i))$ for $(i, j) \in P_2(\beta)$. Note that for $(i \ i')$ and $(j \ j')$ in $I_1(\alpha)$, $\bar{v}(i \ i') = (j \ j')$ and $\bar{v}(i \ i') = (j' \ j)$ are considered as two different permutations of $\{(i \ i'), (j \ j')\}$.

Define

$$\bar{t}_v(\alpha) = \#\{(i \ i') \in I_1(\alpha) \mid \bar{v}(i \ i') = (i' \ i)\}$$

and

$$\bar{t}_v^f(\alpha) = \#\{(i \ i') \in I_1(\alpha) \mid \bar{v}(i \ i') = (i \ i') \text{ and } j_1(v, i) = 0\}.$$

Also define

$$\tilde{t}_v(\beta) = \#\{i \in I_2(\beta) \mid \tilde{v}(i) = j \text{ and } i \neq j\}.$$

Note that $\tilde{t}_v(\beta)$ is always even and $\Omega_{\pm}^w \neq \emptyset$ if and only if $m_k(\alpha)$ and $m_k(\beta)$ are even for each odd part k of α and β .

Define a W_{2n} -module Ξ_n such that the character of Ξ_n is given by

$$\text{tr}(w, \Xi_n) = \left\langle \text{tr}(\cdot, R_{T_w}^1), \text{Ind}_{H^F}^{G^F} \mathbf{1} \right\rangle. \quad (10)$$

Lemma 3.5.

$$\text{tr}(w, \Xi_n) = \sum_{(t_v, v) \in \Omega_{\pm}^w} (-1)^{\ell(\alpha) + \bar{t}_v(\alpha) + \bar{t}_v^f(\alpha) + \frac{\tilde{t}_v(\beta)}{2}}$$

where $(\alpha; \beta)$ is the pair of partitions associated to w .

Proof. We decompose the symplectic space $V = (\mathbb{F}_q)^{4n}$ according to the invariant subspaces of T^F

$$V = \oplus_{(i, i') \in I_1(\alpha)} (V_i \oplus V_{i'}) \oplus \oplus_{i \in I_2(\beta)} V_i,$$

where $T^F|_{V_i \oplus V_{i'}} \cong \mathbb{F}_{q^{\alpha_i}}^\times$ for $(i \ i') \in I_1(\alpha)$ and $T^F|_{V_i} \cong \mathbb{F}_{q^{2\beta_i}}^1$ for $i \in I_2(\beta)$. Fix an element $(t_v, v) \in \Omega_{\pm}^w$ and denote by γ the corresponding representative of the double cosets $T_w^F \backslash \Theta_{T_w}^F / H^F$.

Let $(i \ i')$ be an index in $I_1(\alpha)$ and assume $\bar{v}(i \ i') = (j \ j')$ or $(j' \ j)$. If $i \neq j$, then the restriction of $Z_G(T_w \cap \gamma H \gamma^{-1})$ into the invariant space $V_i \oplus V_j$ or $V_i \oplus V_{j'}$ is isomorphic to $\text{GL}_2(V_i \oplus V_j)$ and its \mathbb{F}_q -rank is 2.

If $i = j$, then α_i is even due to the existence of t_v and we have two cases, $j_1(v, i) \neq 0$ and $j_1(v, i) = 0$. If $\bar{v}(i, i') = (i, i')$ and $j_1(v, i) = 0$, then the restriction of $Z_G(T_w \cap \gamma H \gamma^{-1})$ into the invariant space $V_i \oplus V_{i'}$ is the same as $T^F|_{V_i \oplus V_{i'}} \cong \mathbb{F}_q^{\times \alpha_i}$ and its \mathbb{F}_q -rank is 1. If $\bar{v}(i, i') = (i, i')$ and $j_1(v, i) \neq 0$, the restriction of $Z_G(T_w \cap \gamma H \gamma^{-1})$ is isomorphic to $\text{GL}_2(V_i)$ but invariant under $F^{\frac{\alpha_i}{2}}$ and its \mathbb{F}_q -rank is 2. If $\bar{v}(i, i') = (i', i)$, the restriction of $Z_G(T_w \cap \gamma H \gamma^{-1})$ is isomorphic to $\text{U}_2(V_i \oplus V_{i'})$ and its \mathbb{F}_q -rank is 1.

Let i be an index in $I_2(\beta)$ and $\bar{v}(i, i') = (j, j')$ or (j', j) . If $i \neq j$, the restriction of $Z_G(T_w \cap \gamma H \gamma^{-1})$ into the invariant space $V_i \oplus V_j \oplus V_{j'} \oplus V_{i'}$ is isomorphic to $\text{U}_2(V_i \oplus V_j)$ and its \mathbb{F}_q -rank is 1. If $i = j$, then $j_2(\beta, i) = 0$ and β_i is even due to the existence of t_v . The restriction of $Z_G(T_w \cap \gamma H \gamma^{-1})$ is the same as $T^F|_{V_i} \cong \mathbb{F}_{q^{2\beta_i}}^1$ and its \mathbb{F}_q -rank is 0.

In sum, the \mathbb{F}_q -rank of $Z_G(T_w \cap \gamma H \gamma^{-1})$ and $\bar{t}_v(\alpha) + \bar{t}_v^f(\alpha) + \frac{\bar{t}_v(\beta)}{2}$ have the same parity. By Proposition 2.4, this lemma follows. \square

Recall that $W_{2n} \cong S_{2n} \ltimes (\mathbb{Z}_2)^{2n}$ acts on \mathcal{T}_{2n} . Take the subgroup, isomorphic to $S_n \times S_n$, of W_{2n} consisting of all permutations on the $\{1, 2, \dots, n\} \times \{n+1, \dots, 2n\}$. Let $\sigma_n = \prod_{i=1}^n (i, 2n+1-i)$. Then σ_n in S_{2n} normalizes $S_n \times S_n$, that is, σ_n is the longest Weyl element w_{2n} . Let $K_n = (\langle \sigma_n \rangle \ltimes (S_n \times S_n)) \ltimes (\mathbb{Z}_2)^{2n}$ regarded as a subgroup of W_{2n} and sgn_K be a character of K_n lifting from the non-trivial character of the group $\{e, \sigma_n\}$. Define a virtual module of W_{2n}

$$\kappa_n = \text{Ind}_{K_n}^{W_{2n}} \mathbf{1} - \text{Ind}_{K_n}^{W_{2n}} \text{sgn}_K.$$

Let N_n be the centralizer $Z_{W_{2n}}(\sigma_n)$. Then N_n is isomorphic to $(S_2 \wr S_n) \ltimes (\mathbb{Z}_2)^n$ where $S_2 \wr S_n$ is the wreath product. Define a character sgn_N of N_n by

$$\text{sgn}_N(h) = (-1)^{\#\{h(1), h(2), \dots, h(n)\} \cap \{1', 2', \dots, n'\}}.$$

Let $\nu_n = \text{Ind}_{N_n}^{W_{2n}} \text{sgn}_N$.

A function f on the set \mathcal{P}_{2n} is called *multiplicative* if $f(\alpha; \beta) = \prod_i f(i^{m_i(\alpha)}; 0) f(0; i^{m_i(\beta)})$ where (i^{m_i}) is a partition of $i \cdot m_i$. Now, we decompose κ_n and ν_n into irreducible representations of W_{2n} .

Lemma 3.6.

$$\nu_n = \sum_{\alpha \vdash n} \binom{\alpha}{\alpha}.$$

Moreover, the character function $\text{tr}(\cdot, \nu_n)$ is multiplicative and if w is an element of cycle-type $(\alpha; \beta)$ then the following holds:

- (1) $\text{tr}(w, \nu_n) = 0$ when w is of cycle-type $(i^m; 0)$ or $(0; i^m)$, and i and m are odd;

- (2) $\text{tr}(w, \nu_n) = i^m \frac{(2m)!}{m!}$ when w is of cycle-type $(i^{2m}; 0)$ and i is odd;
- (3) $\text{tr}(w, \nu_n) = (-i)^m \frac{(2m)!}{m!}$ when w is of cycle-type $(0; i^{2m})$ and i is odd;
- (4) $\text{tr}(w, \nu_n) = 0$ when w is of cycle-type $((2i)^m; 0)$ or $(0; (2i)^m)$, and m is odd;
- (5) $\text{tr}(w, \nu_n) = (2i)^{\frac{m}{2}} \frac{m!}{(m/2)!}$ when w is of cycle-type $((2i)^m; 0)$ and m is even;
- (6) $\text{tr}(w, \nu_n) = (-2i)^{\frac{m}{2}} \frac{m!}{(m/2)!}$ when w is of cycle-type $(0; (2i)^m)$ and m is even.

Proof. First, we give a set of representatives for the double cosets $(W_{n_1} \times W_{n_2}) \backslash W_{2n} / N_n$ where $n_1 \leq n_2$ and $n_1 + n_2 = 2n$, which is the same as $S_{n_1} \times S_{n_2} \backslash S_{2n} / S_2 \wr S_n$. Let

$$h_{r,n_1} = \prod_{i=1}^r (n_1 + 1 - i, 2n + 1 - i) \text{ for } 0 \leq r \leq \lfloor \frac{n}{2} \rfloor. \quad (11)$$

Then $\{h_{r,n_1} \mid 0 \leq r \leq \lfloor \frac{n}{2} \rfloor\}$ is a set of representatives for the double cosets $S_{n_1} \times S_{n_2} \backslash S_{2n} / S_2 \wr S_n$. Denote by \bar{h}_{r,n_1} the embedded element in W_{2n} and $\{\bar{h}_{r,n_1} \mid 0 \leq r \leq \lfloor \frac{n}{2} \rfloor\}$ is a set of representatives for the double cosets $W_{n_1} \times W_{n_2} \backslash W_{2n} / N_n$.

Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be the irreducible module of W_{2n} given in Section 2.2. If $|\alpha| \neq |\beta|$, let $n_1 = \min\{|\alpha|, |\beta|\}$. Then $g = (n, n')(n+1, (n+1)')$ commutes with \bar{h}_{r,n_1} for all $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Since $g \in N_n$ and $g = \bar{h}_{r,n_1} g \bar{h}_{r,n_1}^{-1} \in W_{|\alpha|} \times W_{|\beta|}$, we have $\text{sgn}_N(g) = -1$ and $\chi_{|\beta|}(g) = 1$ if $|\alpha| < |\beta|$ or $\bar{\rho}_S(\alpha)(g) = 1$ if $|\alpha| > |\beta|$. By Mackey's theorem,

$$\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \nu_n \rangle_{W_{2n}} = 0 \text{ when } |\alpha| \neq |\beta|.$$

Now, we only need consider the case $|\alpha| = |\beta| = n$. If $r \neq 0$, then $\bar{h}_{r,n} g \bar{h}_{r,n}^{-1} = (2n+1, (2n+1)')(n+1, (n+1)')$ is in $W_{|\beta|}$. Also $\text{sgn}_N(g) = -1$ and $\chi_{|\beta|}(\bar{h}_{r,n} g \bar{h}_{r,n}^{-1}) = 1$. Denote by $N'_n = (W_n \times W_n) \cap N_n$. This is isomorphic to $S_n^\Delta \ltimes (\mathbb{Z}_2)^n$ where S_n^Δ is the subgroup of $S_n \times S_n$, and commutes with σ_n . Using Mackey's theorem and Frobenius reciprocity, we have

$$\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \nu_n \rangle_{W_{2n}} = \langle \rho_S(\alpha) \otimes \rho_S(\beta), \mathbf{1} \rangle_{S_n^\Delta} = \langle \rho_S(\alpha), \rho_S(\beta) \rangle_{S_n}.$$

Next, let us calculate $\text{tr}(w, \nu_n)$ where w is an element of cycle-type $(\alpha; \beta)$. We discuss the values at w in each case separately. If $m_i(\alpha)$ or $m_i(\beta)$ is odd for some odd i , then $Cl_{W_{2n}}(w) \cap N_n = \emptyset$ and $\text{tr}(w, \nu_n) = 0$.

If w is of cycle-type $(i^{2m}; 0)$ or $(0; i^{2m})$ and i is odd, then $Cl_{W_{2n}}(w) \cap N_n$ contains at most one N_n -conjugacy class in N_n . If w is in N_n , then

$$\text{tr}(w, \nu_n) = \frac{|Z_{W_{2n}}(w)|}{|Z_{N_n}(w)|} = \frac{i^{2m}(2m)!2^{2m}}{i^m m! 2^m 2^m} = i^m \frac{(2m)!}{m!}$$

when w is of cycle-type $(i^{2m}; 0)$ and

$$\text{tr}(w, \nu_n) = (-1)^m \frac{|Z_{W_{2n}}(w)|}{|Z_{N_n}(w)|} = \frac{(2i)^{2m}(2m)!}{i^m m! 2^m 2^m} = (-i)^m \frac{(2m)!}{m!}$$

when w is of cycle-type $(0; i^{2m})$.

Let w be an element of cycle-type $(0; (2i)^m)$. If m is odd, then $Cl_{W_{2n}}(w) \cap N_n = \emptyset$. If m is even, then $Cl_{W_{2n}}(w) \cap N_n$ contains at most one N_n -conjugacy class in N_n . If w is in N_n , then

$$\text{tr}(w, \nu_n) = (-1)^{\frac{m}{2}} \frac{Cl_{W_{2n}}(w)}{Cl_{N_n}(w)} = (-1)^{\frac{m}{2}} \frac{(4i)^m m!}{(2i)^{\frac{m}{2}} \frac{m}{2}! 2^{\frac{m}{2}} \cdot 2^{\frac{m}{2}}} = \frac{(-2i)^{\frac{m}{2}} m!}{(m/2)!}.$$

Let w be an element of cycle-type $((2i)^m; 0)$ and $Cl_{W_{2n}}(w) \cap N_n = \emptyset$. In this case, $Cl_{W_{2n}}(w) \cap N_n$ contains $2[\frac{m}{2}] + 1$ conjugacy classes of N_n . By the formula for the induced character,

$$\text{tr}(w, \nu_n) = \frac{Z_{W_{2n}}(w)}{Z_{N_n}(w)} = \frac{(2i)^m m! 2^m}{(2i)^{\frac{m}{2}} \frac{m}{2}! 2^{\frac{m}{2}} \cdot 2^{\frac{m}{2}}} = (2i)^{\frac{m}{2}} \frac{m!}{(m/2)!}$$

when m is even and $\text{tr}(w, \nu_n) = 0$ when m is odd. In addition, it is easy to check that $\text{tr}(\cdot, \kappa_n)$ is multiplicative. This completes the proof. \square

Lemma 3.7.

$$\kappa_n = \sum_{i=0}^n (-1)^i \binom{i, 2n-i}{0}.$$

Moreover, the character $\text{tr}(\cdot, \kappa_n)$ is multiplicative and if w is an element of cycle-type $(\alpha; \beta)$ then

$$\text{tr}(w, \kappa_n) = \begin{cases} 0 & (\alpha; \beta) = ((2i+1)^m; 0) \text{ or } (0; (2i+1)^m) \\ 2^m & (\alpha; \beta) = ((2i)^m; 0) \text{ or } (0; (2i)^m). \end{cases} \quad (12)$$

Proof. Since the characters $\mathbb{1}$ and sgn_K of K_n are trivial on the subgroup $(\mathbb{Z}_2)^{2n}$ of W_{2n} , it is sufficient to show that

$$\text{Ind}_{K'_n}^{S_{2n}} \mathbb{1} - \text{Ind}_{K'_n}^{S_{2n}} \text{sgn}_{K'} = \sum_{i=0}^n (-1)^i \rho_S(i, 2n-i), \quad (13)$$

where $K'_n = \langle \sigma_n \rangle \ltimes (S_n \times S_n)$ is the subgroup $K_n \cap S_{2n}$ and $\text{sgn}_{K'}$ is the restriction of sgn_K on K'_n .

First, applying Pieri's formula, we have

$$\text{Ind}_{S_n \times S_n}^{S_{2n}} \mathbf{1} = \text{Ind}_{K'_n}^{S_{2n}} \mathbf{1} + \text{Ind}_{K'_n}^{S_{2n}} \text{sgn}_{K'} = \sum_{i=0}^n \rho_S(i, 2n-i). \quad (14)$$

Let $g_r = \prod_{i=1}^r (i, 2n+1-i)$ be an element in S_{2n} ($g_r = e$ when $r = 0$, and $g_n = \sigma_n$). Then g_r for $0 \leq r \leq n$ are a complete set of representatives for the double cosets $(S_n \times S_n) \backslash S_{2n} / (S_n \times S_n)$ and $S_{2n} = \coprod_{r=0}^{[n/2]} K'_n g_r K'_n$. Since g_r and σ_n commute for all r , σ_n is in $K'_n \cap g_r K'_n g_r^{-1}$ for each double coset $K'_n g_r K'_n$ in S_{2n} . Since $\mathbf{1}(\sigma_n) \neq \text{sgn}_{K'}(\sigma_n)$, by Mackey's theorem,

$$\left\langle \text{Ind}_{K'_n}^{S_{2n}} \mathbf{1}, \text{Ind}_{K'_n}^{S_{2n}} \text{sgn}_{K'} \right\rangle_{S_{2n}} = 0 \quad (15)$$

and

$$\left\langle \text{Ind}_{K'_n}^{S_{2n}} \mathbf{1}, \text{Ind}_{K'_n}^{S_{2n}} \mathbf{1} \right\rangle_{S_{2n}} = \#K'_n \backslash S_{2n} / K'_n = \left[\frac{n}{2} \right] + 1.$$

Referring to Macdonald [M95, VII (2.4)],

$$\left\langle \text{Ind}_{S_n \times S_n}^{S_{2n}} \mathbf{1}, \text{Ind}_{S_2 \wr S_n}^{S_{2n}} \mathbf{1} \right\rangle_{S_{2n}} = \left[\frac{n}{2} \right] + 1.$$

Also we have

$$\left\langle \text{Ind}_{K'_n}^{S_{2n}} \mathbf{1}, \text{Ind}_{S_2 \wr S_n}^{S_{2n}} \mathbf{1} \right\rangle_{S_{2n}} = \#K'_n \backslash S_{2n} / S_2 \wr S_n = \left[\frac{n}{2} \right] + 1.$$

Hence $\text{Ind}_{K'_n}^{S_{2n}} \mathbf{1}$ is a submodule of $\text{Ind}_{S_2 \wr S_n}^{S_{2n}} \mathbf{1}$. By (14),

$$\text{Ind}_{K'_n}^{S_{2n}} \mathbf{1} = \sum_{i=0}^{[n/2]} \rho_S(2i, 2n-2i).$$

Then Equation (13) follows by (15).

Next, let us calculate $\text{tr}(w, \kappa_n)$ where w is an element of cycle-type $(\alpha; \beta)$. If a part i of α or β is odd, then $Cl_G(w) \cap K_n = Cl_{K_n}(w)$ is in the subgroup $K'_n \rtimes (\mathbb{Z}_2)^{2n}$. By the formula for the induced character, $\text{tr}(w, \kappa_n) = 0$.

If $Cl_{W_{2n}}(w)$ and the left coset $\sigma_n \cdot K'_n \rtimes (\mathbb{Z}_2)^{2n}$ have a non-empty intersection, then $Cl_G(w) \cap K_n$ is the disjoint union of at most two K_n -conjugacy classes. If there are two conjugacy classes, one conjugacy class is in $K'_n \rtimes (\mathbb{Z}_2)^{2n}$ and the other is in $\sigma_n \cdot K'_n \rtimes (\mathbb{Z}_2)^{2n}$. In all cases, we have

$$\text{tr}(w, \kappa_n) = \frac{2|W_{2n}|}{|K_n|} \cdot \frac{|Cl_{W_{2n}}(w) \cap \sigma_n \cdot K'_n \rtimes (\mathbb{Z}_2)^{2n}|}{|Cl_{W_{2n}}(w)|} = \frac{2Z_{W_{2n}}(w)}{Z_{K_n}(w)}.$$

Hence, $\text{tr}(w, \kappa_n)$ is multiplicative and Equation (12) follows since

$$Z_{W_{2n}}(w) = (2i)^m m! 2^m \text{ and } Z_{K_n}(w) = 2 \cdot i^m m! 2^m$$

when $(\alpha; \beta) = ((2i)^m; 0)$ and

$$Z_{W_{2n}}(w) = (4i)^m m! \text{ and } Z_{K_n}(w) = 2 \cdot (2i)^m m!$$

when $(\alpha; \beta) = (0; (2i)^m)$. This completes the proof. \square

Proposition 3.8.

$$\Xi_n = \sum_{r=0}^n \sum_{\beta \vdash n-r} \sum_{i=1}^r (-1)^i \binom{(i, 2r-i) \cdot \beta}{\beta} \quad (16)$$

where $(1^i) \cdot (1^{2r-i}) \cdot \beta$ corresponds to the representation

$$\text{Ind}_{S_{2r} \times S_{n-r}}^{S_{n+r}} \rho_S(i, 2r-i) \otimes \rho_S(\beta).$$

Proof. It is equivalent to show the following identity

$$\Xi_n = \sum_{r=0}^n \text{Ind}_{W_{2r} \times W_{2(n-r)}}^{W_{2n}} \kappa_r \otimes \nu_{n-r}. \quad (17)$$

We will match the characters of the two sides.

Let w be an element in W_{2n} of cycle-type $(\alpha; \beta)$. The set Ω_{\pm}^w may be written as a disjoint union $\coprod_{r=0}^n \Omega_{\pm, r}^w$, where

$$\Omega_{\pm, r}^w = \Omega_{\pm}^w \cap \{v \in W_{2n} \mid \mathcal{S}_v^{(3)} = 2r\}.$$

Define the set

$$\mathcal{P}_r(\alpha; \beta) = \{(a, b) \in \mathcal{P}_{2r} \times \mathcal{P}_{2n-2r} \mid a \cup b = (\alpha; \beta)\}$$

where the union $a \cup b$ is the component-wise union. We may further decompose $\Omega_{\pm, r}^w$ as

$$\Omega_{\pm, r}^w = \coprod_{(a, b) \in \mathcal{P}_r(\alpha; \beta)} \Omega_{\pm, r}^{w'} \times \Omega_{\pm, 0}^{w''}.$$

where w' and w'' are of cycle-types a and b in W_{2r} and W_{2n-2r} respectively. Note that if $(t_{v'}, v')$ is in $\Omega_{\pm, r}^{w'}$, then v' is the identity and we take $t' \in \Omega_{\pm, r}^{w'}$; if $(t_{v''}, v'')$ is in $\Omega_{\pm, 0}^{w''}$ then $t_{v''} = 1$ and we take $v'' \in \Omega_{\pm, 0}^{w''}$. It is easy to observe that $\ell(\alpha) = \ell(\alpha') + \ell(\alpha'')$, $\bar{\ell}_v^f(\alpha) = \ell(\alpha')$, $\bar{\ell}_v(\alpha) = \bar{\ell}_{v''}(\alpha'')$, and $\tilde{\ell}_v(\beta) = \ell(\beta'')$ where $a = (\alpha'; \beta')$ and $b = (\alpha''; \beta'')$. Then

$$\begin{aligned} \text{tr}(w, \Xi_n) &= \sum_{r=0}^n \sum_{(t_v, v) \in \Omega_{\pm, r}^w} \sum_{(a, b) \in \mathcal{P}_r(\alpha; \beta)} \\ &\quad \left(\sum_{t' \in \Omega_{\pm, r}^{w'}} (-1)^{2\ell(\alpha')} \right) \left(\sum_{v'' \in \Omega_{\pm, 0}^{w''}} (-1)^{\ell(\alpha'') + \bar{\ell}_{v''}(\alpha'') + \ell(\beta'')/2} \right). \end{aligned}$$

We consider the characters $\text{tr}(w', \Xi'_r)$ and $\text{tr}(w'', \Xi''_{n-r})$ on W_{2r} and W_{2n-2r} defined by $\text{tr}(w', \Xi'_r) := \#\Omega_{\pm, r}^{w'}$ and

$$\text{tr}(w'', \Xi''_{n-r}) := \sum_{v'' \in \Omega_{\pm, 0}^{w''}} (-1)^{\ell(\alpha'') + \bar{\iota}_{v''}(\alpha'') + \ell(\beta'')/2}.$$

By the definition, the class functions $\text{tr}(w', \Xi'_r)$ and $\text{tr}(w'', \Xi''_{n-r})$ are multiplicative. In order to prove (17), it is enough to show that the characters $\text{tr}(w', \Xi'_r)$ and $\text{tr}(w'', \Xi''_{n-r})$ match $\text{tr}(\cdot, \kappa_{2r})$ and $\text{tr}(\cdot, \nu_{2n-2r})$ on the conjugacy classes of cycle-types $(i^m; 0)$ and $(0; i^m)$. This is easily verified by Lemma 3.7 and Lemma 3.6 and the definition of $\text{tr}(\cdot, \Xi'_r)$ and $\text{tr}(\cdot, \Xi''_{n-r})$. Then this lemma follows. \square

4. DISTINGUISHED SYMBOLS

In this last section, we will use the Littlewood-Richardson rule to decompose Ξ_n into irreducible representations of W_{2n} and then divide those constituents into a sum of virtual cells. Doing so, we conclude that there is a bijection between symbols in those cells and the distinguished unipotent representations.

The set-theoretic difference of the Young diagrams of a pair of partitions $(\alpha; \beta)$ is called a skew diagram of shape α/β and size $|\alpha/\beta|$ is $|\alpha| - |\beta|$. Let $(\alpha/\beta)_i = \alpha_i - \beta_i$. A skew diagram is a *horizontal strip* (resp. *vertical strip*) if it contains at most one box in each column (resp. row). For a skew diagram η , let $h(\eta)$ be the horizontal strip obtained by removing all columns from η which contain more than one box and $v(\eta)$ be the skew diagram obtained by removing $h(\eta)$ from η . For example,

$$\eta = \begin{array}{ccccc} & & \square & \square & \square \\ & & \square & \square & \\ \square & \square & & & \\ \square & & & & \end{array} \quad h(\eta) = \begin{array}{cc} \square & \square \end{array} \quad v(\eta) = \begin{array}{cc} \square & \square \\ \square & \square \\ \square & \end{array}.$$

Denote by $|h(\eta)|$ (resp. $|v(\eta)|$) the number of boxes of $h(\eta)$ (resp. $v(\eta)$).

A horizontal strip η is called *even* if the number of boxes in each row of η is even. Let Γ_2 be the set of all skew diagrams which contain at most 2 boxes in each column and is of even size, and Γ_2° be a subset of Γ_2 of all skew diagrams η such that $h(\eta)$ is even.

Let η be in Γ_2 . Define a *tableau* T of shape η by putting the integers '1' or '2' in each box of η such that the follow Littlewood-Richardson condition holds: The integers in all boxes are listed from right to left then from top to bottom, then at the first k entries in this list for each $1 \leq k \leq |\eta|$, each '1' occurs at least as many times as '2'.

Define $\text{Tab}(\eta)$ to be the set of all tableaux of shape η and $\text{Tab}(\eta)_i$ to be the subset of $\text{Tab}(\eta)$ consisting of tableaux of i ‘2’s. For example, when $i > \frac{|\eta|}{2}$, $\text{Tab}(\eta)_i = \emptyset$ and $|\text{Tab}(\eta)_i| = 0$. By convention, if $|\eta| = 0$, define $\text{Tab}(\eta) = \{\emptyset\}$ and $|\text{Tab}(\eta)| = 1$, where $|\text{Tab}(\eta)|$ (resp. $|\text{Tab}(\eta)_i|$) is the size of the set $\text{Tab}(\eta)$ (resp. $\text{Tab}(\eta)_i$). Also we have the disjoint union $\text{Tab}(\eta) = \cup_{i \geq 0} \text{Tab}(\eta)_i$.

Next, we use the Littlewood-Richardson rule to decompose Ξ_n into irreducibles and get the following lemma.

Lemma 4.1.

$$\Xi_n = \sum_{\substack{(\alpha; \beta) \in \mathcal{P}_{2n} \\ \alpha/\beta \in \Gamma_2^\circ}} (-1)^{\frac{|v(\alpha/\beta)|}{2}} \binom{\alpha}{\beta}.$$

Proof. By Equation (16), if $\langle \Xi_n, \binom{\alpha}{\beta} \rangle \neq 0$, then $\beta \leq \alpha$ and $\alpha/\beta \in \Gamma_2$. If $\alpha = \beta$, then α/β is in Γ_2° and $\langle \Xi_n, \binom{\alpha}{\beta} \rangle = 1$.

If $\beta < \alpha$, by the Littlewood-Richardson rule,

$$\langle \Xi_n, \binom{\alpha}{\beta} \rangle = \sum_{i=0}^{|\alpha/\beta|/2} (-1)^i |\text{Tab}(\alpha/\beta)_i| = \sum_{i=|v(\alpha/\beta)|/2}^{|\alpha/\beta|/2} (-1)^i |\text{Tab}(\alpha/\beta)_i|.$$

Since $|\text{Tab}(\alpha/\beta)_i| = |\text{Tab}(h(\alpha/\beta))_{i-|v(\alpha/\beta)|/2}|$, we have

$$\langle \Xi_n, \binom{\alpha}{\beta} \rangle = (-1)^{|v(\alpha/\beta)|/2} \sum_{i=0}^{|h(\alpha/\beta)|/2} (-1)^i |\text{Tab}(h(\alpha/\beta))_i|.$$

For a tableau T in $\text{Tab}(\eta)$, let $|T_2|$ be the number of boxes filled with 2’s. For a subset \mathcal{T} of $\text{Tab}(\eta)$, define

$$\zeta(\mathcal{T}) = \sum_{T \in \mathcal{T}} (-1)^{|T_2|}. \quad (18)$$

Next, we will show that for a horizontal strip η of even size

$$\zeta(\text{Tab}(\eta)) = \begin{cases} 1 & \text{if } \eta \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

Let $\mathcal{T}(\eta)_{(i)}$ be the subset of $\text{Tab}(\eta)$ consisting of all tableaux whose first box filled with ‘2’ occurs at the right-end box of the i -th row. By the Littlewood-Richardson condition, $\mathcal{T}(\eta)_{(1)} = \emptyset$. Denote by $\mathcal{T}(\eta)_{\ell(\eta)+1}$ the subset consisting of the one tableau whose boxes all contain ‘1’, where $\ell(\eta)$ is the number of rows of η . We have a disjoint union of

$\text{Tab}(\eta)$ and a formula of $\zeta(\text{Tab}(\eta))$:

$$\text{Tab}(\eta) = \prod_{i=2}^{\ell(\eta)+1} \mathcal{T}(\eta)_{(i)} \text{ and } \zeta(\text{Tab}(\eta)) = 1 + \sum_{i=2}^{\ell(\eta)} \zeta(\mathcal{T}(\eta)_{(i)}).$$

In order to prove (19), we only need to consider the rows whose numbers of boxes are non-zero.

In order to evaluate $\zeta(\mathcal{T}(\eta)_{(i)})$ for $i \geq 2$, we reduce to a skew diagram $\eta^{(i)}$ of a smaller size than η . Let $\eta^{(i)}$ be the horizontal strip such that the numbers of rows are $(\sum_{j=1}^{i-1} \eta_j - 1, \eta_i - 1, \dots, \eta_{\ell(\eta)})$ from top to bottom, obtained by merging the boxes in the top $i-1$ rows of η to the i -th row, becoming the first row of $\sum_{j=1}^{i-1} \eta_j$ boxes, and then removing the right end boxes in the top two rows. For instance,

$$\eta = \begin{array}{c} \boxed{1} \boxed{1} \\ \boxed{1} \boxed{1} \\ \boxed{1} \boxed{2} \end{array} \quad \eta^{(2)} = \begin{array}{c} \boxed{1} \boxed{1} \boxed{1} \\ \boxed{1} \end{array}.$$

Then we obtain a bijection between $\mathcal{T}(\eta)_{(i)}$ and $\text{Tab}(\eta^{(i)})$, and

$$\zeta(\mathcal{T}(\eta)_{(i)}) = -\zeta(\text{Tab}(\eta^{(i)})).$$

Now, we have

$$\zeta(\text{Tab}(\eta)) = 1 - \sum_{i=2}^{\ell(\eta)} \zeta(\text{Tab}(\eta^{(i)})) \quad (20)$$

and $|\eta^{(i)}| = |\eta| - 2$ for $2 \leq i \leq \ell(\eta)$. We apply this inductive formula (20) to prove (19). If $|\eta| = 0$, $\zeta(\text{Tab}(\eta)) = 1$. If $|\eta| = 2$, we have two types of skew shapes, $\boxed{}\boxed{}$ and $\boxed{}\boxed{}$, denoted by $(2)/(0)$ and $(2,1)/(1)$ respectively. Then it is easy to check that $\zeta(\text{Tab}((2)/(0))) = 1$ and $\zeta(\text{Tab}((2,1)/(1))) = 0$.

In general, if η is even, then $\eta^{(i)}$ is not even for all $2 \leq i \leq \ell(\eta)$ and $|\eta^{(i)}| < |\eta|$. By induction, $\zeta(\text{Tab}(\eta^{(i)})) = 0$ and then $\zeta(\text{Tab}(\eta)) = 1$ by (20). If instead η is not even, let i_{\max} be the maximal integer such that $\eta_{i_{\max}}$ is odd. Then $\eta^{(i)}$ is even if and only if $i = i_{\max}$. By induction, we have $\zeta(\text{Tab}(\eta^i)) = 1$ when $i = i_{\max}$ and $\zeta(\text{Tab}(\eta^i)) = 0$ when $i \neq i_{\max}$. Hence $\zeta(\text{Tab}(\eta)) = 1 - \zeta(\text{Tab}(\eta^{(i_{\max})})) = 0$. This completes the proof of (19) and the lemma follows. \square

For example, when $G = \mathrm{Sp}_4$, $\Xi_1 = \bar{\rho}_S(2) - \bar{\rho}_S(1^2) + \binom{1}{1}$. If $G = \mathrm{Sp}_{12}$, then

$$\begin{aligned} \Xi_3 = & \binom{3}{3} + \binom{1,2}{1,2} + \binom{1^3}{1^3} + \binom{4}{2} + \binom{2^2}{2} - \binom{1^2,2}{2} + \binom{1,3}{1^2} \\ & - \binom{2^2}{1^2} - \binom{1^4}{1^2} + \binom{5}{1} - \binom{1^2,3}{1} + \binom{1,2^2}{1}. \end{aligned}$$

Let \mathcal{S}_n be a set consisting of all special symbols of rank $2n$ whose associated pairs of partitions under the map \mathcal{L}^{-1} are even horizontal strips, that is, $\mathcal{L}^{-1}(\mathcal{S}_n)$ is the same as the subset of Γ_2° consisting of all horizontal strips of size $2n$. Given a special symbol

$$Z = \begin{pmatrix} \lambda_1, \lambda_2, \dots, \lambda_{m+1} \\ \mu_1, \mu_2, \dots, \mu_m \end{pmatrix},$$

define an admissible arrangement

$$\Phi_Z = \{(\lambda_i, \mu_i) \mid \lambda_i \neq \mu_i, \text{ for } 1 \leq i \leq m\}$$

and a subset of Φ_Z

$$\hat{\Phi}_Z = \{(\lambda_i, \mu_i) \in \Phi \mid \lambda_i - \mu_i \equiv 1 \pmod{2}\}.$$

Theorem 4.2.

$$\mathcal{U}(\mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})}^{\mathrm{Sp}_{4n}(\mathbb{F}_q)} \mathbb{1}) = \sum_{Z \in \mathcal{S}_n} R(c(Z, \Phi_Z, \hat{\Phi}_Z)).$$

In particular, the unipotent cuspidal representation of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$ has non-trivial $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ -invariants.

Proof. First, we prove that

$$\Xi_n = \sum_{Z \in \mathcal{S}_n} c(Z, \Phi, \hat{\Phi}_Z). \quad (21)$$

Assume that α/β is in Γ_2° and $|\alpha| + |\beta| = 2n$. Then $\ell(\alpha) - \ell(\beta) \leq 2$.

Recall that we increase the lengths of α and β such that $\ell(\alpha) = \ell(\beta) + 1$ by adding zeros. Assume that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{m+1})$ and $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ with $\alpha_i \leq \alpha_{i+1}$ and $\beta_i \leq \beta_{i+1}$ and at least one of α_1 and β_1 is nonzero.

Since $\beta < \alpha$, we have $\beta_i \leq \alpha_{i+1}$ for $1 \leq i \leq m$. Set $\lambda_i = \alpha_i + i - 1$ for $1 \leq i \leq m+1$ and $\mu_i = \beta_i + i - 1$ for $1 \leq i \leq m$. Then the symbol $\Lambda = \binom{\lambda}{\mu}$ is of rank $2n$ and defect 1. Since $\beta < \alpha$, Λ is special if and only if $\alpha_i \leq \beta_i$ for $1 \leq i \leq m$. Then the skew diagram α/β has no two boxes in each column, that is, α/β is a horizontal strip and even. Hence, Λ is special if and only if Λ is in \mathcal{S}_n .

Let Ψ be a subset of Φ_Z . For each subset $\Psi \subset \Phi_Z$, define

$$\Lambda(\Psi) = \left(\begin{array}{c} Z_0 \coprod \Psi_* \coprod (\Phi - \Psi)^* \\ Z_0 \coprod \Psi^* \coprod (\Phi - \Psi)_* \end{array} \right)$$

and denote by $\mathcal{L}^{-1}(\Lambda(\Psi)) = \left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix} \right)$. Since $\max\{\alpha_i, \beta_i\} \leq \min\{\alpha_{i+1}, \beta_{i+1}\}$ for $1 \leq i \leq m$ (when $i = m$, let $\min\{\alpha_{i+1}, \beta_{i+1}\} = \alpha_{m+1}$), we have $\beta'_i \leq \alpha'_{i+1}$ for $1 \leq i \leq m$.

We will show that for all subsets Ψ of Φ_Z

$$\left\langle \Xi_n, (-1)^{|\Psi \cap \hat{\Phi}_Z|} \mathcal{L}^{-1}(\Lambda(\Psi)) \right\rangle = 1. \quad (22)$$

If $\Psi = \emptyset$, then $\Lambda(\Psi) = Z$ and Equation (22) is true. We verify Equation (22) by induction. Assume that Equation (22) is true for Ψ . Then by adding a pair (λ_i, μ_i) in $\hat{\Phi}_Z$ but not in Ψ , we obtain a subset $\Psi_1 = \Psi \cup \{(\lambda_i, \mu_i)\}$.

By $(\lambda_i, \mu_i) \notin \Psi$, we may assume

$$\mathcal{L}^{-1}(\Lambda(\Psi)) = \left(\begin{array}{c} \alpha'_1, \alpha'_2, \dots, \alpha_i, \alpha'_{i+1}, \dots, \alpha'_m, \alpha_{m+1} \\ \beta'_1, \beta'_2, \dots, \beta'_{i-1}, \beta_i, \dots, \beta'_m \end{array} \right)$$

and then

$$\mathcal{L}^{-1}(\Lambda(\Psi_1)) = \left(\begin{array}{c} \alpha'_1, \alpha'_2, \dots, \beta_i, \alpha'_{i+1}, \dots, \alpha_{m+1} \\ \beta'_1, \beta'_2, \dots, \beta'_{i-1}, \alpha_i, \dots, \beta'_m \end{array} \right).$$

Now let us consider the skew diagrams obtained by the differences of partitions in $\mathcal{L}^{-1}(\Lambda(\Psi))$ and $\mathcal{L}^{-1}(\Lambda(\Psi_1))$ respectively, which are the same except the following two rows

$$\mathcal{L}^{-1}(\Lambda(\Psi)): \begin{array}{c} \beta'_{i-1} \quad \alpha_i - \beta'_{i-1} \quad \beta_i - \alpha_i \quad \boxed{\alpha'_{i+1} - \beta_i} \\ \beta'_{i-1} \quad \boxed{\alpha_i - \beta'_{i-1}} \end{array}$$

and

$$\mathcal{L}^{-1}(\Lambda(\Psi_1)): \begin{array}{c} \beta'_{i-1} \quad \alpha_i - \beta'_{i-1} \quad \boxed{\beta_i - \alpha_i} \quad \boxed{\alpha'_{i+1} - \beta_i} \\ \beta'_{i-1} \quad \boxed{\alpha_i - \beta'_{i-1}} \quad \boxed{\beta_i - \alpha_i} \end{array}.$$

Here the boxes are the two rows of the skew diagrams associated with $\mathcal{L}^{-1}(\Lambda(\Psi))$ and $\mathcal{L}^{-1}(\Lambda(\Psi_1))$, and the integers in the boxes are the parts corresponding to the differences of partitions in $\mathcal{L}^{-1}(\Lambda(\Psi))$ and $\mathcal{L}^{-1}(\Lambda(\Psi_1))$. The integers on the left of the boxes are the parts of the smaller partitions in $\mathcal{L}^{-1}(\Lambda(\Psi))$ and $\mathcal{L}^{-1}(\Lambda(\Psi_1))$. By the assumption on $\mathcal{L}^{-1}(\Lambda(\Psi))$ and $\alpha_i < \beta_i$, the skew diagram of $\mathcal{L}^{-1}(\Lambda(\Psi_1))$ is also in Γ_2° and

$$|v(\mathcal{L}^{-1}(\Lambda(\Psi_1)))| = |v(\mathcal{L}^{-1}(\Lambda(\Psi)))| + 2(\beta_i - \alpha_i).$$

Then by Lemma 4.1

$$\left\langle \Xi_n, \mathcal{L}^{-1}(\Lambda(\Psi_1)) \right\rangle_{W_{2n}} = (-1)^{|\Psi \cap \hat{\Phi}_Z| + \beta_i - \alpha_i}.$$

In addition, by the definition of Φ_Z , $(-1)^{|\Psi_1 \cap \hat{\Phi}_Z|} = (-1)^{|\Psi \cap \hat{\Phi}_Z| + \beta_i - \alpha_i}$. Therefore

$$\langle \Xi_n, \mathcal{L}^{-1}(\Lambda(\Psi_1)) \rangle_{W_{2n}} = (-1)^{|\Psi_1 \cap \hat{\Phi}_Z|}.$$

On the other hand, if there is a partition $(\alpha'_{\beta'})$ in Γ_2° , one may reverse the previous operation (i.e. removing pairs) and obtain a partition in $\mathcal{L}^{-1}(\mathcal{S}_n)$.

In sum, $\sum_{Z \in \mathcal{S}_n} c(Z, \Phi_Z, \hat{\Phi}_Z)$ is a summand of Ξ_n , and every irreducible W_{2n} -module in Ξ_n is in $\sum_{Z \in \mathcal{S}_n} c(Z, \Phi, \hat{\Phi}_Z)$ with the same signature. Then Equation (21) follows.

Let d be the number of singles in Z . By $\langle \Xi_n, c(Z, \Phi_Z, \hat{\Phi}_Z) \rangle_{W_{2n}} = 2^d$,

$$\langle R(c(Z, \Phi_Z, \hat{\Phi}_Z)), \text{Ind}_{H^F}^{G^F} \mathbb{1} \rangle_{G^F} = 2^d.$$

In addition, $\langle R(c(Z, \Phi_Z, \hat{\Phi}_Z)), R(c(Z, \Phi_Z, \hat{\Phi}_Z)) \rangle_{G^F} = 2^d$. By Theorem 1.1, every unipotent representation in $R(c(Z, \Phi_Z, \hat{\Phi}_Z))$ is distinguished. This completes the theorem. \square

Example 4.3. Let $G^F = \text{Sp}_4(\mathbb{F}_q)$. We continue Example 2.2. In this case, $\mathcal{S}_1 = \left\{ \binom{2}{-}, \binom{0,2}{1} \right\}$. Let $Z = \binom{0,2}{1}$ and $\Phi_Z = \{(0, 1)\}$. By (2) and Theorem 4.2,

$$\mathcal{U}(\text{Ind}_{\text{Sp}_{2n}(\mathbb{F}_{q^2})}^{\text{Sp}_{4n}(\mathbb{F}_q)} \mathbb{1}) = \mathbb{1} + \rho \binom{0,1}{2} + \theta_{10}.$$

Example 4.4. Let $G^F = \text{Sp}_{12}$ and

$$\mathcal{S}_6 = \left\{ \binom{0,4}{3}, \binom{0,2,4}{1,3}, \binom{0,2,3,4}{1,2,3}, \binom{0,5}{2}, \binom{2,3}{2}, \binom{0,2,5}{1,2}, \binom{0,6}{1}, \binom{6}{-} \right\}.$$

Let $Z = \binom{0,5}{2}$ and $\Phi_Z = \{(0, 2)\}$ and $\hat{\Phi}_Z = \emptyset$. We have

$$R(c(Z, \Phi_Z, \emptyset)) = R \binom{0,5}{2} + R \binom{2,5}{0} = \rho \binom{0,5}{2} + \rho \binom{0,2}{5}.$$

Hence these two unipotent representations are distinguished.

If $Z = \binom{0,2,4}{1,3}$ then $\Phi_Z = \{(0, 1), (2, 3)\}$ and $\hat{\Phi}_Z = \Phi_Z$. We have

$$R(c(Z, \Phi_Z, \Phi_Z)) = R \binom{0,2,4}{1,3} - R \binom{1,2,4}{0,3} - R \binom{0,3,4}{1,2} + R \binom{1,3,4}{0,2}.$$

The decomposition of $R(c(Z, \Phi_Z, \Phi_Z))$ is given in (3) and each constituent is distinguished.

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